JOURNAL OF COMPUTATIONAL PHYSICS 105, 333–338 (1993)

A Consistent Formulation of the Anisotropic Stress Tensor for Use in Models of the Large-Scale Ocean Circulation

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Received November 6, 1991; revised June 11, 1992

Subgrid-scale dissipation of momentum in numerical models of the large-scale ocean circulation is commonly parameterized as a viscous diffusion resulting from the divergence of a stress tensor of the form $\boldsymbol{\sigma} = \mathbf{A}:\nabla\mathbf{u}$. The form of the fourth-order coefficient tensor A is derived for anisotropic dissipation with an axis of rotational symmetry. Sufficient conditions for A to be positive definite for incompressible flows, so guaranteeing a net positive dissipation of kinetic energy, are found. The divergence of the stress tensor, in Cartesian and spherical polar coordinates, is given for A with constant and spatially varying elements. A consistent form of A and σ for use in models based on the Arakawa B- and C-grids is also derived.

1. INTRODUCTION

The effect of Reynolds' stresses in the oceans, or scales unresolved by the numerical grid in general circulation models (GCMs) of the large-scale ocean circulation, is commonly parameterized in terms of an *eddy* viscosity, $\nabla\cdot\boldsymbol{\sigma} = \nabla\cdot(\mathbf{A}:\nabla\mathbf{u})$, where A is a fourth-order tensor. In modelling the large-scale ocean circulation, A is usually chosen to be anisotropic with symmetry about the radial coordinate, and the magnitude of the elements is assumed independent of position in the fluid. For some applications, however, a formulation in which the elements of A are spatially varying is appropriate; for example, if variable grid resolution is used in a boundary layer, if "sponge" layers are artificially created to absorb wave energy, or if some Richardson number dependent formulation is used. GCMs cast in spherical polar coordinates are now so widely used that is it often forgotten that the form of the viscous metric terms in the horizontal momentum equations arises because A is assumed anisotropic as described in [1, 2]. The correct formulation as an exact divergence in the momentum equations is important as it guarantees that the basin will be a net sink of kinetic energy for all flows, provided A is positive definite. Proof of uniqueness of solution to the linearized system is based on a dissipation theorem, which also requires A be positive definite. Further, if the elements of A are spatially varying, then care must be taken to retain the terms which ensure that pure rotation does not generate a viscous stress. The form of the stress tensor for a fluid with tranverse isotropy is re-examined in Section 2 and the exact form of its divergence if the elements of A are non-constant is given in Cartesian and spherical polar coordinates. An appropriate finite-difference formulation of the divergence for use in numerical ocean GCMs, based on those developed by Bryan and Cox at GFDL, Princeton [3, 4], and now widely used for climate modelling, e.g., [5, 6], is presented and discussed in Section 3.

2. VISCOUS STRESS IN ORTHOGONAL CURVILINEAR COORDINATES

The basic equations of motion for a non-rotating Boussinesq fluid subject to a viscous stress are

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho_0}\nabla(p - \rho g z) + \nabla\cdot\boldsymbol{\sigma}. \tag{1}$$

For a Newtonian fluid, the deviatoric stress tensor, $\boldsymbol{\sigma}$, is assumed linearly dependent on the velocity gradients $\nabla\mathbf{u}$,

$$\boldsymbol{\sigma} = \mathbf{A}:\nabla\mathbf{u}, \tag{2a}$$

where A is a fourth-order tensor, whose elements depend on the local state of the fluid, but not directly on the velocity distribution, see, e.g., [7]. In the commonly adopted parameterization of subgrid-scale processes, the turbulent stresses are assumed dependent on the large-scale velocity gradients as in (2), although more complex forms of A are often used, e.g., [5].

Symmetry of $\boldsymbol{\sigma}$, i.e., $\sigma_{ij} = \sigma_{ji}$, requires a symmetry in the fourth-order tensor A, i.e., $A_{ijkl} = A_{jikl}$. Further, it is usually assumed that no viscous stress is generated in a fluid as a result of pure rotation, and so $A_{ijkl} = A_{ijlk}$. Therefore (2a) reduces to

$$\boldsymbol{\sigma} = \mathbf{A}:\mathbf{e}, \tag{2b}$$

0021-9991/93 $5.00

where $\mathbf{e}=\frac{1}{2}(\nabla\mathbf{u}+[\nabla\mathbf{u}]^T)$, and so $\mathbf{A}$ has 36 independent elements. It is also assumed that there exists an energy functional $\mathscr{D}=\mathbf{e}:\mathbf{A}:\mathbf{e}$, which is second-order differentiable in $\mathbf{e}$. Hence $\partial^2\mathscr{D}/(\partial e_{ij}\partial e_{kl})=\partial^2\mathscr{D}/(\partial e_{kl}\partial e_{ij})$, and so $A_{ijkl}=A_{klij}$, which reduces the number of independent elements to 21.

The rate of change of kinetic energy of a parcel of fluid of volume V bounded by a surface S is

$$\begin{aligned}\frac{D}{Dt}\int_V \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\,dV &= \int_V \{-\rho_0^{-1}\mathbf{u}\cdot\nabla(p-\rho gz)+\mathbf{u}\cdot\nabla\cdot(\mathbf{A}:\mathbf{e})\}\,dV \\ &= \int_S \{-\rho_0^{-1}(p-\rho gz)\mathbf{u}+\mathbf{u}\cdot\mathbf{A}:\mathbf{e}\}\cdot d\mathbf{S} \\ &\quad -\int_V \mathbf{e}:\mathbf{A}:\mathbf{e}\,dV,\end{aligned}$$

on applying the divergence theorem and assuming that the fluid is incompressible ($\nabla\cdot\mathbf{u}=0$). Therefore, if $\mathbf{A}$ is positive definite in the sense that for any second-order symmetric tensor $\mathbf{e}$, $\mathscr{D}=\mathbf{e}:\mathbf{A}:\mathbf{e}>0$, then there is no internal source of kinetic energy.

(a) *General Form*

If we consider a system of orthogonal curvilinear coordinates ξ_1, ξ_2, ξ_3 with unit vectors $\hat{\boldsymbol{\xi}}_1, \hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\xi}}_3$ parallel to the coordinate lines and in the direction of increasing value, then change in the position vector $\mathbf{x}$ corresponding to increments in ξ_1, ξ_2, ξ_3 is

$$\delta\mathbf{x}=h_1\,\delta\xi_1\,\hat{\boldsymbol{\xi}}_1+h_2\,\delta\xi_2\,\hat{\boldsymbol{\xi}}_2+h_3\,\delta\xi_3\,\hat{\boldsymbol{\xi}}_3,$$

and the divergence of the stress tensor is given by

$$\begin{aligned}\hat{\boldsymbol{\xi}}_1\cdot(\nabla\cdot\boldsymbol{\sigma})=&\frac{1}{h_1h_2h_3}\{(h_2h_3\sigma_{11})_{,1}+(h_3h_1\sigma_{12})_{,2} \\ &+(h_1h_2\sigma_{13})_{,3}\}+\frac{\sigma_{12}}{h_1h_2}h_{1,2}+\frac{\sigma_{13}}{h_1h_3}h_{1,3} \\ &-\frac{\sigma_{22}}{h_1h_2}h_{2,1}-\frac{\sigma_{33}}{h_1h_3}h_{3,1},\end{aligned}\tag{3}$$

where $(\ )_{,i}$ denotes the partial derivative with respect to ξ_i. Similar expressions for the $\hat{\boldsymbol{\xi}}_2, \hat{\boldsymbol{\xi}}_3$ components are given by cyclically rotating the indices. The strain components are

$$\begin{aligned}e_{11}&=\frac{u_{1,1}}{h_1}+\frac{u_2}{h_1h_2}h_{1,2}+\frac{u_3}{h_3h_1}h_{1,3}, \\ e_{22}&=\frac{u_{2,2}}{h_2}+\frac{u_3}{h_2h_3}h_{2,3}+\frac{u_1}{h_1h_2}h_{2,1} \\ e_{33}&=\frac{u_{3,3}}{h_3}+\frac{u_1}{h_3h_1}h_{3,1}+\frac{u_2}{h_2h_3}h_{3,2} \\ 2e_{23}&=\frac{h_3}{h_2}\left(\frac{u_3}{h_3}\right)_{,2}+\frac{h_2}{h_3}\left(\frac{u_2}{h_2}\right)_{,3}, \\ 2e_{31}&=\frac{h_1}{h_3}\left(\frac{u_1}{h_1}\right)_{,3}+\frac{h_3}{h_1}\left(\frac{u_3}{h_3}\right)_{,1}, \\ 2e_{12}&=\frac{h_2}{h_1}\left(\frac{u_2}{h_2}\right)_{,1}+\frac{h_1}{h_2}\left(\frac{u_1}{h_1}\right)_{,2}.\end{aligned}\tag{4}$$

The above expressions may be found in standard mathematical texts, e.g., [8]. The incompressibility condition $\nabla\cdot\mathbf{u}=0$ is just $\operatorname{tr}\mathbf{e}=0$.

(i) *Transverse Isotropy.* If there is an axis of symmetry about which the fluid is isotropic, assume that it is the ξ_3 axis; then the only non-zero elements of A_{ijkl} are A_{iijj}, $A_{ijij_{i\neq j}}$ given by

$$A_{iijj}\equiv a_{ij}=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{12} & a_{11} & a_{13}\\ a_{13} & a_{13} & a_{33}\end{pmatrix}\tag{5a}$$

and

$$\begin{aligned}&A_{1212}=\tfrac{1}{2}(a_{11}-a_{12}), \qquad A_{2323}=\tfrac{1}{2}a_{44}, \\ &A_{3131}=\tfrac{1}{2}a_{44},\end{aligned}\tag{5b}$$

and so the number of independent coefficients reduces to five. A method for determining the above is given in standard texts on theoretical elasticity, e.g., [8, 9]. As expressed in (1), $\boldsymbol{\sigma}$ is the deviatoric stress tensor, and so $\operatorname{tr}\boldsymbol{\sigma}=0$, which requires

$$a_{33}=a_{11}+a_{12}-a_{13}.\tag{5c}$$

Hence the stress tensor for an incompressible fluid, assuming an axis of rotational symmetry, may be expressed in the coordinate invariant form

$$\begin{aligned}&\sigma_{11}=(A_M+\nu)\,e_{11}+(\nu-A_M)\,e_{22}, \\ &\sigma_{22}=(\nu-A_M)\,e_{11}+(\nu+A_M)\,e_{22}, \\ &\sigma_{33}=2\nu e_{33}, \\ &\sigma_{12}=2A_Me_{12}, \qquad \sigma_{23}=2\kappa_Me_{23}, \qquad \sigma_{31}=2\kappa_Me_{31},\end{aligned}\tag{6}$$

where

$$\begin{aligned}&A_M\equiv\frac{1}{2}(a_{11}-a_{12}), \qquad \kappa_M\equiv\frac{a_{44}}{2}, \\ &\nu\equiv\frac{1}{2}(a_{11}+a_{12})-a_{13}.\end{aligned}\tag{7}$$

The coefficients A_M, κ_M may be identified with the usual horizontal and vertical eddy viscosity coefficients. Typical values of A_M and κ_M used in ocean models are order 10^7 and $1\ \mathrm{cm}^2\,\mathrm{s}^{-1}$, respectively. A physical interpretation of ν and possible values is discussed in [2] and is addressed later in part (b).

The dissipation rate may also be expressed in invariant form

$$\mathscr{D} = \mathbf{e}:\mathbf{A}:\mathbf{e} = A_M(e_{11}-e_{22})^2 + \tfrac{3}{2}\nu(e_{11}+e_{22})^2 + 4\kappa_M(e_{23}^2+e_{31}^2) + 4A_M e_{12}^2, \tag{8}$$

and so sufficient conditions for $\mathbf{A}$ to be positive definite ($\mathscr{D}>0$, for any $\mathbf{e}$) are

$$A_M > 0, \qquad \kappa_M > 0, \qquad \nu > 0. \tag{9}$$

(ii) *3-D Isotropy.* If the fluid is wholly isotropic, then $a_{33}=a_{11}$, $a_{12}=a_{13}$, and $a_{44}=(a_{11}-a_{12})/2$ in (5), and so the number of independent elements of $\mathbf{A}$ reduces to two. In terms of the viscosity coefficients, for an isotropic fluid $A_M=\kappa_M=\nu$.

In applications to modelling the large-scale ocean circulation, Eq. (5) is used with either constant values or with values dependent on the Richardson number, e.g., [5]. Most observational and experimental studies of the ocean circulation have been concerned with dissipation due to vertical diffusion and have used the expression $(\kappa(\mathbf{u})_{,z})_{,z}$ as their basis, e.g., [10, 11]. In the remainder of this section, a consistent formulation of viscous stress, in which the elements of $\mathbf{A}$ are spatially varying, is derived for an incompressible fluid in the two most commonly used coordinate systems.

(b) *Cartesian Coordinates*: (x_1, x_2, x_3), $(h_1, h_2, h_3) = (1, 1, 1)$

(i) *Transverse Isotropy with Constant Coefficients.* Substituting (5) in (2b) and (3) yields

$$\mathbf{i}\cdot(\nabla\cdot\boldsymbol{\sigma}) = A_M(u_{1,11}+u_{1,22}) + \kappa_M u_{1,33} \;[+(\kappa_M-\nu)\,u_{3,31}], \tag{10a}$$

with a corresponding expression for the $\mathbf{j}$ component. The vertical component is

$$\mathbf{k}\cdot(\nabla\cdot\boldsymbol{\sigma}) = \kappa_M(u_{3,11}+u_{3,22}) + (2\nu-\kappa_M)\,u_{3,33}. \tag{10b}$$

The physical basis for considering transverse isotropy is that the vertical scales of motion are much smaller than the horizontal scales in the ocean. If L, d are typical horizontal and vertical scales, then the last term in (10a), enclosed by [ ], is negligible in comparison with the previous if $d^2/L^2 \ll 1$, and $\nu \ll \min(A_M,\ \kappa_M L^2/d^2)$. The large-scale ocean circulation is also assumed to be in hydrostatic balance, and so the terms in (10b) must be negligible in comparison with the acceleration due to gravity, g. Hence, we require $\kappa_M U/(gLd) \ll 1$ and $\nu U/(gLd) \ll 1$, where U is a typical horizontal velocity scale. Typical values are $U \sim 1\ \mathrm{m\,s^{-1}}$, $L \sim 10$ km, $d \sim 10$ m, $g \simeq 10\ \mathrm{m\,s^{-2}}$, and so the vertical momentum equation reduces to hydrostatic balance if κ_M, $\nu \ll 10^6\ \mathrm{m^2\,s^{-1}}$. Values of ν varying from 0 to A_M are suggested by [2] for motions ranging from highly convective in the vertical to fairly stable, as for the oceans. However, for a consistent shallow-water description with hydrostatic balance, ν does not enter the equations of motion, provided it is small enough.

(ii) *Spatially Varying Coefficients.* Maintaining the dependence derived for transverse isotropy, but now letting the elements of $\mathbf{A}$ vary spatially, yields the following expression for the $\mathbf{i}$-component of the divergence of the stress

$$\begin{aligned}\mathbf{i}\cdot\nabla\cdot\boldsymbol{\sigma} = {} & \{A_M u_{1,1}\}_{,1} + \{A_M u_{1,2}\}_{,2} + \{\kappa_M u_{1,3}\}_{,3} \\ & [+(\kappa_M-\nu)\,u_{3,31}] \\ & [+\underline{(\kappa_M)_{,3}\,u_{3,1} - (\nu)_{,1}\,u_{3,3}}] \\ & + \underline{(A_M)_{,2}\,u_{2,1} - (A_M)_{,1}\,u_{2,2}}\end{aligned} \tag{11a}$$

with a corresponding expression for the $\mathbf{j}$ component. The vertical component is

$$\begin{aligned}\mathbf{k}\cdot\nabla\cdot\boldsymbol{\sigma} = {} & (\kappa_M u_{3,1})_{,1} + (\kappa_M u_{3,2})_{,2} \\ & + \{(2\nu-\kappa_M)\,u_{3,3}\}_{,3} \\ & + \underline{\kappa_{M,1}u_{1,3} + \kappa_{M,2}u_{2,3} + \kappa_{M,3}u_{3,3}}.\end{aligned} \tag{11b}$$

Equation (11a) is somewhat different from the form occasionally quoted in which the underlined terms are missing, but it is the strictly correct form if pure rotation does not generate a viscous stress. It may be argued that pure rotation will not occur in the spin-up of an ocean GCM, but it is a free solution of the problem and so it could be generated as a result of rounding errors. It is worth emphasizing that the underlined terms in (11a) arise because the viscous diffusion of momentum results from the divergence of a stress tensor. In describing the diffusion of heat with variable diffusion coefficients, no such terms would arise.

Once again, under the shallow-water approximation, the [ ]-terms in (11a) may be neglected, and if κ_M, $\nu \ll (gLd)/U$, (11b) may be ignored.

(c) *Spherical Polar Coordinates*: (λ, ϕ, r), $(h_1, h_2, h_3) = (r\cos\phi, r, 1)$

The coordinate system in chosen to coincide with that used in ocean modelling, i.e., λ is the longitude, ϕ is the latitude, and r is the radial coordinate. The horizontal components of the divergence of the stress tensor, as given by (3), are

$$\begin{aligned}\hat{\boldsymbol{\lambda}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= \sigma_{r\lambda,r}+\frac{1}{r}\sigma_{\phi\lambda,\phi}+\frac{1}{r\cos\phi}\sigma_{\lambda\lambda,\lambda}\\ &\quad+\frac{1}{r}(3\sigma_{r\lambda}-2\sigma_{\lambda\phi}\tan\phi)\\ \hat{\boldsymbol{\phi}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= \sigma_{r\phi,r}+\frac{1}{r}\sigma_{\phi\phi,\phi}+\frac{1}{r\cos\phi}\sigma_{\phi\lambda,\lambda}\\ &\quad+\frac{1}{r}((\sigma_{\lambda\lambda}-\sigma_{\phi\phi})\tan\phi+3\sigma_{r\phi})\end{aligned} \tag{12}$$

and the strain field, as given by (4), is

$$\begin{aligned} e_{\lambda\lambda} &= \frac{1}{r\cos\phi}u_{\lambda,\lambda}-\frac{1}{r}u_\phi\tan\phi+\frac{u_r}{r},\\ e_{\phi\phi} &= \frac{1}{r}u_{\phi,\phi}+\frac{u_r}{r}, \qquad e_{rr}=u_{r,r},\\ 2e_{\lambda\phi} &= \frac{\cos\phi}{r}\left(\frac{u_\lambda}{\cos\phi}\right)_{,\phi}+\frac{1}{r\cos\phi}u_{\phi,\lambda},\\ 2e_{r\lambda} &= u_{\lambda,r}-\frac{u_\lambda}{r}+\frac{u_{r,\lambda}}{r\cos\phi},\\ 2e_{r\phi} &= u_{\phi,r}-\frac{u_\phi}{r}+\frac{u_{r,\phi}}{r}.\end{aligned} \tag{13}$$

(i) *Transverse Isotropy with Constant Coefficients.* If the fluid has an axis of isotropic symmetry coinciding with the r axis, then **A** is given by (5), and so combining with (13) and substituting in (12) yields

$$\begin{aligned}\hat{\boldsymbol{\lambda}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= A_M\left\{\frac{u_{\lambda,\lambda\lambda}}{r^2\cos^2\phi}+\frac{(\cos\phi\, u_{\lambda,\phi})_{,\phi}}{r^2\cos\phi}\right.\\ &\quad\left.+\frac{(1-\tan^2\phi)}{r^2}u_\lambda-\frac{2\sin\phi}{r^2\cos^2\phi}u_{\phi,\lambda}\right\}+\kappa_M u_{\lambda,rr}\\ &\quad\left[+2\kappa_M\left\{\frac{u_{r,\lambda}}{r^2\cos\phi}+\frac{u_{\lambda,r}}{r}-\frac{u_\lambda}{r^2}\right\}\right.\\ &\quad\left.+(\kappa_M-\nu)\frac{u_{r,r\lambda}}{r\cos\phi}\right]\\ \hat{\boldsymbol{\phi}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= A_M\left\{\frac{u_{\phi,\lambda\lambda}}{r^2\cos^2\phi}+\frac{(\cos\phi\, u_{\phi,\phi})_{,\phi}}{r^2\cos\phi}\right.\\ &\quad\left.+\frac{(1-\tan^2\phi)}{r^2}u_\phi+\frac{2\sin\phi}{r^2\cos^2\phi}u_{\lambda,\lambda}\right\}+\kappa_M u_{\phi,rr}\\ &\quad\left[+2\kappa_M\left\{\frac{u_{r,\phi}}{r^2}+\frac{u_{\phi,r}}{r}-\frac{u_\phi}{r^2}\right\}\right.\\ &\quad\left.+(\kappa_M-\nu)\frac{u_{r,r\phi}}{r}\right],\end{aligned} \tag{14}$$

where A_M, κ_M, ν are as defined in (7). In ocean models, the thin shell approximation, $|r| \ll R$, the Earth's radius, is made. This results in the factors r^{-1} in (14) being replaced by R^{-1} and $\partial/\partial r$ being replaced by $\partial/\partial z$. Further, the [ ]-terms may be neglected under the shallow-water approximation.

(ii) *Spatially Varying Coefficients.* If the elements of **A** vary spatially, then (14) is modified to

$$\begin{aligned}\hat{\boldsymbol{\lambda}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= \frac{(A_M u_{\lambda,\lambda})_{,\lambda}}{r^2\cos^2\phi}+\frac{(A_M\cos\phi\, u_{\lambda,\phi})_{,\phi}}{r^2\cos\phi}\\ &\quad+\frac{(1-\tan^2\phi)}{r^2}A_M u_\lambda-\frac{2\sin\phi}{r^2\cos^2\phi}A_M u_{\phi,\lambda}\\ &\quad+\left\{\frac{\tan\phi}{r^2}u_\lambda+\frac{u_{\phi,\lambda}}{r^2\cos\phi}\right\}(A_M)_{,\phi}\\ &\quad-\left\{\frac{u_{\phi,\phi}}{r^2\cos\phi}+\frac{\sin\phi}{r^2\cos^2\phi}u_\phi\right\}(A_M)_{,\lambda}+(\kappa_M u_{\lambda,r})_{,r}\\ &\quad\left[+2\kappa_M\left\{\frac{u_{r,\lambda}}{r^2\cos\phi}+\frac{u_{\lambda,r}}{r}-\frac{u_\lambda}{r^2}\right\}\right.\\ &\quad\left.+(\kappa_M-\nu)\frac{u_{r,r\lambda}}{r\cos\phi}\right]\\ &\quad\left[+\left\{\frac{u_{r,\lambda}}{r\cos\phi}-\frac{u_\lambda}{r}\right\}(\kappa_M)_{,r}-\frac{u_{r,r}}{r\cos\phi}(\nu)_{,\lambda}\right]\\ \hat{\boldsymbol{\phi}}\cdot(\nabla\cdot\boldsymbol{\sigma}) &= \frac{(A_M u_{\phi,\lambda})_{,\lambda}}{r^2\cos^2\phi}+\frac{(A_M\cos\phi\, u_{\phi,\phi})_{,\phi}}{r^2\cos\phi}\\ &\quad+\frac{(1-\tan^2\phi)}{r^2}A_M u_\phi+\frac{2\sin\phi}{r^2\cos^2\phi}A_M u_{\lambda,\lambda}\\ &\quad+\left\{\frac{\tan\phi}{r^2}u_\phi-\frac{u_{\lambda,\lambda}}{r^2\cos\phi}\right\}(A_M)_{,\phi}\\ &\quad+\left\{\frac{\sin\phi}{r^2\cos^2\phi}u_\lambda+\frac{u_{\lambda,\phi}}{r^2\cos\phi}\right\}(A_M)_{,\lambda}+(\kappa_M u_{\phi,r})_{,r}\\ &\quad\left[+2\kappa_M\left\{\frac{u_{r,\phi}}{r^2}+\frac{u_{\phi,r}}{r}-\frac{u_\phi}{r^2}\right\}\right.\\ &\quad\left.+(\kappa_M-\nu)\frac{u_{r,r\phi}}{r}\right]\\ &\quad\left[+\left\{\frac{u_{r,\phi}}{r}-\frac{u_\phi}{r}\right\}(\kappa_M)_{,r}-\frac{u_{r,r}}{r}(\nu)_{,\phi}\right].\end{aligned} \tag{15}$$

Once again, the explicitly written gradient terms for A_M, κ_M are required since pure rotations do not generate a stress, and so **A** must be symmetric in its third and fourth indices. Under the shallow-water approximation, the [ ]-terms may be neglected. A similar set of equations was proposed by [12] and adopted by [13, 14]. The formulation was slightly different, which in the present notation is equivalent to setting $\nu = 0$ in (6) and making the shallow-water approximation, and so neglecting the [ ]-terms in (15).

3. A CONSISTENT FINITE-DIFFERENCE FORMULATION

Although early papers on finite-difference models of the large-scale ocean circulation specify the divergence of the stress tensor in terms of the symmetric components of strain, the actual numerical formulation in [4] is like (15), but with the explicit gradient terms missing as A_M, κ_M are assumed constant. For simplicity and brevity, possible formulations of the above are described for Cartesian finite-difference grids of the form typically used in ocean models, see [15, 16].

(i) *Arakawa B-grid.*

The grid is staggered with u, v specified at corners of a rectangle in the horizontal plane with w at the centre. In the vertical plane w-points lie between u, v points. The incompressibility condition may be expressed as

$$\delta_x \bar{u}^y + \delta_y \bar{v}^x + \delta_z w = 0,$$

where $\delta_x(\cdot)$, $\overline{(\cdot)}^x$ denote centered-differencing and -averaging, respectively, and are given by

$$\begin{aligned}(\delta_x u)_{i+1/2,j,k} &= \frac{1}{\Delta_{i+1/2}} (u_{i+1,j,k} - u_{i,j,k}) \\ (\delta_x w)_{i,j,k+1/2} &= \frac{1}{\Delta_i} (w_{i+1/2,j,k+1/2} \\ &\quad - w_{i-1/2,j,k+1/2}) \\ \overline{(u)_{i+1/2,j,k}}^x &= \frac{1}{2} (u_{i+1,j,k} + u_{i,j,k}),\end{aligned} \tag{16}$$

where $\Delta_{i+1/2} = (\Delta_{i+1} + \Delta_i)/2$, with similar definitions for y, z. Only grid-spacings which vary in the direcion of progression are considered. Therefore Δx is a function of i, but not of j, k, and the operators δ_x, δ_y, and δ_z are interchangeable, and similarly for the averaging operators. However, the operators δ_x and $\overline{(\ )}^x$ are not interchangeable if the grid-spacing is variable, and so the finite-difference equivalent of $\nabla_H \times (\nabla_H p)$ is not identically zero as it is for constant grid-spacing. The remainder is $O(\max\{(\Delta x)^2, (\Delta y)^2\})$, which is the same as the accuracy of the centered-differencing scheme. The strain components are

$$e_{xx} = \delta_x \bar{u}^y, \qquad e_{yy} = \delta_y \bar{v}^x, \qquad e_{zz} = \delta_z w,$$
$$2e_{xy} = \delta_y \bar{u}^x + \delta_x \bar{v}^y, \qquad 2e_{xz} = \delta_z u + \delta_x \bar{w}^y,$$
$$2e_{yz} = \delta_z v + \delta_y \bar{w}^x.$$

The stress components will be of the form, using (6),

$$\sigma_{xx} = A_M e_{xx} - A_M e_{yy} - \bar{\nu}^z e_{zz},$$
$$\sigma_{xy} = 2A_M e_{xy}, \qquad \sigma_{xz} = 2\overline{\kappa_M}^{xy} e_{xz},$$

where A_M is defined at w-points in the horizontal and u, v-points in the vertical, and κ_M, ν are defined at w-points in both the horizontal and the vertical. The **i**-component of the divergence of the stress tensor will be of the form

$$\mathbf{i} \cdot (\nabla \cdot \boldsymbol{\sigma}) = \delta_x \overline{\sigma_{xx}}^y + \delta_y \overline{\sigma_{xy}}^x + \delta_z \sigma_{xz}.$$

The terms corresponding to the underlined terms in (11a) will come from

$$VA = \delta_y \overline{(A_M \delta_x \bar{v}^y)}^x - \delta_x \overline{(A_M \delta_y \bar{v}^x)}^y$$
$$VK = \delta_z (\overline{\kappa_M}^{xy} \delta_x \bar{w}^y) - \delta_x \overline{(\bar{\nu}^z \delta_z w)}^y.$$

Ideally, the finite-difference scheme should be formulated such that there is exact cancelation of the terms of the type $A_M \delta_x \delta_y v$ in VA, and of the type $\kappa_M \delta_x \delta_z w$, if $\nu = \kappa_M$, in VK. For the B-grid, if the grid-spacing is variable, there is not exact cancelation. If the grid spacing is constant, there is exact cancelation in VA, yielding

$$VA = \overline{\delta_y A_M \delta_x v}^x - \overline{\delta_x A_M \delta_y v}^y,$$

where the identity $\delta_y \overline{(a \delta_x \bar{b}^y)}^x = \overline{a \delta_{xy} b}^{xy} + \overline{\delta_y a \delta_x b}^x$ has been used. For VK, there is exact cancelation for variable grid-spacing provided κ_M, ν are not functions of x, y, yielding

$$VK = \delta_z \kappa_M \delta_x \bar{w}^y + \overline{(\kappa_M - \nu)}^z \delta_{xz} \bar{w}^y.$$

If κ_M is a function of x, y, then there is cancelation up to an error of the order of the square of the maximum grid-spacing for both constant and variable grid-spacings. Of course, under the shallow-water approximation, the VK term may be neglected with the diffusion due a variable κ_M just resulting from $(\kappa_M \mathbf{u}_{,z})_{,z}$.

The above presentation is a formal derivation of the viscous terms on a B-grid. In practice, e.g., [4], the double averaging is not performed, and terms of the form $A_M \delta_x \overline{(\delta_x \overline{(u)}^y)}^y$ are represented as $A_M \delta_{xx} u$.

(ii) *Arakawa C-grid.* The u, v, w-grids are all staggered in the horizontal, forming a cross with the w-point at the center, v-points to the north and south, and u-points to the east and west. In the vertical, the u, v-points are at the same depth with the w-points in between. The incompressibility condition may be expressed as

$$\delta_x u + \delta_y v + \delta_z w = 0,$$

where the centered-differencing and -averaging operators are defined similarly to (16). For the C-grid with variable grid-spacing, the finite-difference equivalent of $\nabla_H \times (\nabla_H p)$ is identically zero. The strain components are

$$e_{xx} = \delta_x u, \qquad e_{yy} = \delta_y v, \qquad e_{zz} = \delta_z w,$$
$$2e_{xy} = \delta_y u + \delta_x v, \qquad 2e_{xz} = \delta_z u + \delta_x w,$$
$$2e_{yz} = \delta_z v + \delta_y w.$$

The stress components will be of the form

$$\sigma_{xx} = \overline{A_M}^y e_{xx} - \overline{A_M}^y e_{yy} - \bar{v}^z e_{zz},$$

$$\sigma_{xy} = 2\overline{A_M}^x e_{xy}, \qquad \sigma_{xz} = 2\overline{\kappa_M}^x e_{xz},$$

where A_M is defined at v-grid points, and κ_M, v at w-grid points. The divergence of the stress tensor is defined by terms of the form

$$\mathbf{i} \cdot (\nabla \cdot \boldsymbol{\sigma}) = \delta_x \sigma_{xx} + \delta_y \sigma_{xy} + \delta_z \sigma_{xz}.$$

The terms corresponding to the underlined terms in (11a) come from

$$VA = \delta_y(\overline{A_M}^x \delta_x v) - \delta_x(\overline{A_M}^y \delta_y v)$$
$$= \delta_y \overline{A_M}^x \delta_x \bar{v}^y - \delta_x \overline{A_M}^y \delta_y \bar{v}^x$$

$$VK = \delta_z(\overline{\kappa_M}^x \delta_x w) - \delta_x(\bar{v}^z \delta_z w)$$
$$= \delta_z \overline{\kappa_M}^x \delta_x \bar{w}^z - \delta_x \overline{\kappa_M}^z \delta_z \bar{w}^x$$
$$+ \overline{(\kappa_M - v)}^{xz} \delta_{xz} w,$$

where the identity $\delta_x(\bar{a}^y \delta_y b) \equiv \delta_x \bar{a}^y \delta_y \bar{b}^x + \bar{a}^{xy} \delta_{xy} b$ has been applied, and the above is valid for variable as well as constant grid-spacing.

4. SUMMARY

The form of the viscous stress tensor used in models of the large-scale ocean circulation has been examined. A formulation of the stress tensor for a transversely isotropic fluid for which the elements of **A** may vary spatially has been proposed. Its formulation is consistent with the requirement that pure rotation does not produce a stress in the fluid, and it permits easy confirmation, if **A** varies spatially, that the viscous stress will act as a sink of kinetic energy within the fluid for all continuous flows. The resulting extra terms in the momentum equations, expressed in Cartesian and spherical polar coordinates, have been calculated, and the terms that are negligible under the shallow-water and hydrostatic approximations are identified. Also, consistent finite-difference schemes for the two most commonly used grids in ocean models have been presented.

ACKNOWLEDGMENTS

During the course of this work, the author was supported by a Japan Society for Promotion of Science Fellowship and thanks Professor K. Takeuchi and colleagues for their hospitality.

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